Swami Ramanand Teerth Marathwada University,
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"Fractional Differential Equations and its Numerical Solution using Simulation Technique"

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Abstract:

We discuss standard approaches to the problem of fractional derivatives and fractional integrals (simply called differintegrals), namely the Riemann-Liouville, the Caputo and the sequential approaches. We prove the basic properties of differ integrals including the rules for their compositions and the conditions for the equivalence of various definitions. Further, we give a brief survey of the basic methods for the solving of linear fractional differential equations and mention limits of their usability. In particular, we formulate the theorem describing the structure of the initial-value problem for linear two-term equations. Finally, we consider some physical applications, in particular fractional advection- dispersion equation and the viscoelasticity problem. Dealing with the first issue we derive the fundamental solution in the form of Levy α-stable distribution density and then we discuss relations between the generalized entral limit theorem and the choice of the corresponding fractional model. In section concerning viscoelasticity we mention some typical models generalizing the standard ones via the replacement of the classical terms by the fractional terms. In particular, we derive the step response functions of those generalized systems.

Introduction:

The subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to its development. They included Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville. Ravi P.Agarwal, Mouffak Benchohra, and Samira Hamani. (2009). Studied .The sufficient conditions are established for the existence of Solutions for a class of boundary value problems for fractional differential equations involving the Caputo fractional derivative. Ibtisam K. Hanan.(2011). used the homotopy analysis method to solve special types of the initial value problems that consist of multi-fractional order integro differential equation in which the fractional derivative and fractional integral in them are described in the Caputo sense and Riemann-Liouville sense respectively. K. Malar, P. Karthikeyan and R. Arul.(2012). Studied the existence and uniqueness

of solution for nonlocal Impulsive fractional Integro Differential equations involving the Caputo fractional derivative in a Banach Space. Anatoly A. Kilbas, Natalia V. Zhukovskaya, Studied the linear nonhomogeneous ordinary differential equations with three left-hand sided Lioville derivatives of fractional order. Rudolf Hilfer, YuryLuchko, Zivorad Tomovski.(2009), used the operational calculus to solve the corresponding initial value problem for the general n-term linear equation with these generalized fractional derivatives of arbitrary orders and types with constant coefficients.Yang Xiao-Jun.(2010).Studied the fractional trigonometric functions in complex-valued space and proposes a short outline of local fractional calculus of complex function in fractal spaces. Amera Almusharrf. (2011). developed fractional trigonometry and generalize Wronskian determinant that can be used to determine the linear independence of a set of solutions to a system of fractional differential equations. We also introduce a new set of fractional differential equations whose solutions are fractional trigonometric functions. This synopsis **"**Fractional Differential Equations and its Numerical Solution using Simulation Technique**"** based on the definitions of fractional derivative and integration, also on special functions as Gamma function, Beta function, Mittag-Leffler Function, and miller – Ross function.

Definition of the Riemann Liouville Fractional Integral:

The Riemann–Liouville definition. For α ∈[n−1,n), the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

Definition of the Caputo's Fractional Derivative :

Caputo definition. For α ∈ [n−1,n), the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\int_a^{\alpha} \frac{f^{(n)}(x)}{(\mathrm{t}-\mathrm{x})^{\alpha-\mathrm{n}+1}}dx.$$

Simulation Technique

1) **The Legendre Pseudospectral Method:**

The Legendre Pseudospectral method is a way of writing any problem modeled with differential equations as a problem with only algebraic constraints by use of the

Pseudospectral differential matrix.

2) Chebyshev Wavelets:

The well-known chebyshev polynomials of 1st kind $T_m(.)$ are defined by a simple recurrence formulation

$$T_m(x) = 2xT_{m-1}(x) - T_{m-1}(x),$$

Assuming that $T_0(x) \triangleq 1$ and $T_1(x) \triangleq x$.

3) Cubic B-Spline Wavelet Collocation method:

The fractional B-Splines are define by

$$B_\alpha(t) = \frac{\Delta^{\alpha+1}t^\alpha}{\Gamma(\alpha+1)}, \quad \alpha > -1$$

Where $\Delta^\alpha f(t) = \sum_{k\geq 0}(-1)^k \binom{\alpha}{k} f(t-k), \ \alpha > 0$

4) Novel Numerical Method:

Suppose the two dimensional Laplacian has a complete set of orthonormal eigen functions $\varphi_{n,m}$ corresponding to eigen values $\lambda_{n,m}$ in a rectangular region $D = \{(x,y)|0 \leq x \leq a, 0 \leq y \leq b\}$, $B(\varphi) = 0$ on ∂D, where $B(\varphi)$ is the homogeneous dirichlet boundary condition

5) Homotopy Analysis Method

A homotopy between two continuous function $f(x)\ and\ g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function $\mathcal{H}: X * [0,1] \rightarrow Y$ from the product of the space X with the unit interval [0,1] toY such that , it x in X then $\mathcal{H}(x;0)$=f(x) and $\mathcal{H}(x;1) = g(x)$.

Review of literature and development in the subject area:

1) Brief history:

The fractional calculus is a natural extension of the traditional calculus. It is like many other mathematical branches and ideas; it has its origin in the pursuing for the extension of meaning. Well known examples are the extension of the integer numbers to the rational numbers, of the rational numbers to the real numbers, and of the real numbers to the complex numbers. The question ofextension of meaning in differential

and integral calculus: Can the derivative$\frac{d^n f}{dx^n}$ of integer order n > 0, be extended to n any order fractional, irrational or complex ? The answer of this question has led to the development of a new theory which is called fractional calculus . The concept of fractional calculus is believed to have emerged from a question raised in the year 1695 by Marquis de L.Hôpital. In a letter dated September 30th, 1695 L.Hôpital wrote to Leibniz asking him about a particular notation he had used in his publications for the n-th derivative $\frac{d^n f(x)}{dx^n}$ of the linear function $f(x) = x$. L.Hôpital posed the question to Leibniz, what would be the result be if n = 0, Leibniz responded to the question, that "$d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx:x}$" .

2) Recent Works:

On the other side, many of researchers considers fractional differential equations, S. Z. Rida and A. A. M. Arafa developed a new application of the Mittag-Leffler Function method that will extend the application of the method to linear differential equations with fractional order. A new solution is constructed in power series. The fractional derivatives are described in the Caputo sense. To illustrate the reliability of the method, some examples are provided. The results reveal that the technique introduced here is very effective and convenient for solving linear differential equations of fractional order.

Recently, a new iterative method was presented by Daftardar – Gejji and Jafari. This technique solves many types of non linear equations such as ordinary and partial differential equations of integer and fractional order. Jafari et al applied. This method to obtain the solutionof linear /non-linear diffusion and wave fractional equations. Daftardar – Gejji and Bhalekar used it for solving fractional boundary value problems and evolution equations.

Numerical methods and analysis of the fractional order partial differential equations are limited to date. Some different numerical methods for solving the space or time fractional partial differential equations have been proposed Liu et al transformed the space fractional partial differential equation in to a system of ordinary differential equations (Method of Lines) which was then solved using backward differentiation formulae.

Shenet al proposed an explicit finite difference approximation for the space fractional diffusion equation the stability and convergence of the method is discussed Lin and Liu proposed the high order approximation of the fractional ordinary differential equation and discussed the convergence and stability of the fractional high order methods.

R. K. Saxena, R. K. Yadav, S. D. Purohit and S. L. Kalla (2005)

In this work authors evaluated Kober fractional basic integral operator of the basic analogue of the H-function. The results obtained are believed to be a new contribution to the theory of q-fractional calculus and are likely to find certain applications to the solution of the fractional q-integral equations involving various q-hypergeometric functions.

H. J. Haubold, A. M. Mathai and R. K. Saxena (2009),

Motivated essentially by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, the authors present in a unified manner, a detailed account or rather a brief survey of the Mittag-Leffler function, generalized Mittag-Leffler functions, Mittag-Leffler type functions, and their interesting and useful properties. Applications of Mittag-Leffler functions in certain areas of physical and applied sciences are also demonstrated. In this survey paper, nearly all types of Mittag-Leffler type functions existing in the literature are presented. An attempt is made to present nearly an exhaustive list of references concerning the Mittag-Leffler functions to make the reader familiar with the present trend of research in Mittag-Leffler type functions and their applications.

V. B. L. Chaurasia and Y. Singh (2011),

Authors established an integral concerning the product of generalized Lauricella function and two H-function of several complex variables using the series representation of H-function. Further, by giving suitable values to the parameters, the main integral reduces to F function.

I. Importance of theStudy:

Fractional calculus is becoming a useful and key tool in the analysis of scientific problems in Physics, Engineering, Biology and Economics, Fractional formalism to be useful in solving practical problems, fractional calculus, integration

and differentiation of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concept of rate of change and area under thecurve.

a) International Status:

Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel, Liouvielli, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache etc.

b) National Status:

Indian's also contributed remarkable in the development of fractional calculus like R.K. Sexana, V.Srivastava, K.N. Rai, S. Das, K. Gupta, S. Saha Roy, V.B.L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna etc.

c) Regional Status:

Many researchers from Maharashtra have been actively involve in the study of fractional calculus. Like Kiran, M. Kolwankar, A.K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, K.C. Takale, J. Verma, D.B. Dhaigudeetc.

d) Scope of Study:

The analysis of Fractional Differential Equations and its Numerical Solution using Simulation Technique. During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics. There are few different methods for obtaining the solution of fractional differential equation (FDE), therefore, we develop new different methods to obtain their solutions and discuss the stability and convergence of the numerical methods upto the length. We will study the powerful mathematical software Mathematica to simulate the solutions graphically. Further we will study fractional wave equations, fractional differential equations and fractional partial differential equations. We will develop iterative and finite difference schemes to solve them. Solutions of these equations will be simulated graphically. We intend to publish the work in reputed

Journals.

II. Objectives of Proposed ResearchWork:

1) To study linear and nonlinear ordinary and partial differentialequations.

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2) to study basic concept of fractionalcalculus.

3) To study analytical and numerical methods for solving linear and nonlinear fractional partial differential equations.

4) To study of Development in Fractional Derivatives, Fractional differential Equations, to Methods of their Solution and some of their Applications

5) To obtain analytical solutions of fractional differential equations. we obtain the approximate solutions of fractional partial differential equations by developed fractional order finite difference schemes and compare these solutions with their analytical solution, if itis possible.

6) To discussconvergence and stability of solutions obtained by developed numerical methods.

7) To stimulate the solutions of fractional partial differential equations using Mathematical software Mathematica to discuss the nature of the solutions.

8) To study the recent applications of **"**Fractional Differential Equations and its Numerical Solution using Simulation Technique**"** and fractional partial differential equations in a various fields ofscience and technology.

9) To publish research papers in reputed journals.

III. Hypothesis:

The proposed work is to prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering field.

Hence, in the present work, we will try to develop the new iterative techniques for fractional differential equation and to obtain their solution. These will represented graphically by Mathematica software.

IV. Limitations of thework:

1) The fractional order model are more adequate than integer order model.
2) Fractional derivatives and integrals enable the description of memory properties of various materials andprocesses.
3) Most of the fractional differential equations do not have analytical solutions.

V. ResearchMethodology:

The following steps will be including in the methodology.

1) To refer the relevant reference books on partial differential equation and fractional partial differential equation.
2) To study Mathematicasoftware.
3) To study fractional calculus.
4) To study Iterative / Numerical methods for obtaining approximate solutions of fractional differential equations.
5) To discuss the stability and convergence of Iterative / Numerical methods.
6) To represent Numerical solutions of fractional differential equations graphically by Mathematica and discuss theirnature.

VI. Expectedoutcome:

1) We will develop Solvability of a Fractional Boundary Value Problem We will develop numerical methods for fractional partial differentialequations.
2) To find the approximate solutions of fractional differential equations by developed new iterative technique and it will represented graphically by Mathematica software.

VII. Year wise plan of work and targets to beachieved:

First year plan:

1) To refer the reference books on differential equation and fractional differential equation.
2) To develop different methods, Numerical methods for differentialequation.
3) To obtain the solutions of partial differential equations by iterative methods.

4) To visualize solutions by graphs using a powerful software Mathematicaand MATLAB.

Second year plan:

1) To study fractionalcalculus.
2) To study fractional differentialequations.
3) To study of Development in Fractional Derivatives, Fractional differential Equations, to Methods of their Solution and some of their Applications
4) To visualize the solutions by graphs using Mathematica, MATLAB and discuss The nature.
5) To discuss the stability and convergence ofmethods.

Third year plan:

1) To develop Solvability of a Fractional Boundary Value Problem.
2) To discuss stability and convergence of the numericalmethods.
3) We will publish our work in reputedjournals.
4) Submission of completed work to concernedauthority.

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